A beneficial role of government bonds without the cash-in-advance constraint*

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Abstract

I study a matching model of money to show that the existence of bonds can be beneficial to a society, compared to having only money. In the model, anonymous agents randomly meet in pairs to produce and consume, hence money becomes essential. I compare two identical economies except the availability of bonds, in the sense that people can use both money and bonds as payments when bonds are available. Following the mechanism design approach, I define implementable allocations and the optimum. Under the notion of the implementability, social planner can devise trading mechanisms that induce people to hold both assets without exogenously given advantages of money as means of payment. The result shows that having both bonds and money in the economy can improve social welfare over having only money. I argue that this role of bonds is associated with a beneficial effect of inflation produced by lump-sum transfers, and it is different from Kocherlakota (2003).

Keywords: Money, Higher Return Assets, Coexistence, Matching Model

JEL classification: E40, E42

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1 Introduction

In the well-known Hicks (1935), he asks why an individual chooses to hold money rather than interest bearing assets, and asserts that this is "the central issue in the pure theory of money". Putting it differently, if there are two types of risk-free assets that can potentially be used as payments for trades, and one of them earns more interest than the other, why does one hold an asset with lower interest? Once we deal with this issue, it naturally leads us to ask: Is it desirable for a society to have both money and higher return assets? In particular, given that nominal risk-free bonds do not provide additional risk-sharing over money, what additional benefit can such bonds provide over money?

The goal of this paper is to examine this question in a monetary economy in which coexistence of money and interest bearing assets (Hicks question) is achieved endogenously. I study a monetary economy in which people cannot borrow and lend, and cannot be taxed, due to frictions such as complete anonymity. Even under this extreme circumstance, social planner can circulate government bonds, whose return is higher than money, by financing the interest through money creation. The model builds on Deviatov (2006), where he studies a model related to Shi (1995) and Trejos and Wright (1995) with an augmented set of money holdings. Following the mechanism design approach in Monetary Economics (Wallace (2010)), he considers the set of implementable allocations and finds numerical examples that some degree of money creation (and inflation) is welfare improving. In this paper, I add an interest bearing asset (inflation-indexed bond) to that economy as a representative of government bonds, and show that having this asset can enhance social welfare.

The coexistence is achieved in the spirit of Zhu and Wallace (2007), where they show that there exists a trading mechanism under which 1) trade outcomes lie in pairwise core and 2) both money and bonds are held by people. Specifically, the trading mechanism divides gains from trade depending on the proportion of money in consumers' portfolios, using the multiplicity of pairwise core allocations. Hence, agents hold some amount of money and forgo interest from bonds for the sake of the gains from trade. Under the information structure and thet notion of defection that I assume, Zhu and Wallace (2007) trading mechanism is included to what social planner can choose. Consequently, the coexistence can be achieved without relying on any assumed advantages of money as a payment.

After laying out the environment and definition of implementability, I study the numerical examples that are adopted from Deviatov (2006), where he considers a same model with only money. I consider a richer set of transfer scheme than what he considers. Computation result shows that there are better allocations which are implementable with money and bonds, compared to the optima with only money. Compared to the optima with only money, transfer and inflation rate are higher in the better allocations. The result indicates that the beneficial effect of inflation produced by lump-sum transfers, as in Deviatov (2006); Green and Zhou (2005); Kehoe et al. (1992); Molico (2006), can be achieved more effectively by the existence of bonds.

Although this paper primarily aims to discover a role of government bonds, it also sheds some light on the welfare implication of currency substitution. When the inflation rate of a country is positive, the currency of that country loses its value over time. Suppose that agents in that country have an access to foreign currency, which is issued in a foreign country where inflation rate is zero. In that case, foreign currency is equivalent to bonds considered in this model. If social planner cannot force people to use only domestic currency, agents may hold and use foreign currency (currency substitution). Hence, the result can also be viewed as an implication of currency substitution.

1.1 Related Literature

In this subsection, I review two papers in which the coexistence is necessary to achieve desirable allocations. For the literature that attempt to rationalize the coexistence, I refer readers to Lagos (2013) and Hu and Rocheteau (2013).

Kocherlakota (2003) is the first normative analysis of the coexistence of money and higher return assets. He argues that the existence of illiquid bonds (illiquid because they cannot be used as payment for consumption goods) can improve social welfare, while liquid bonds cannot. The coexistence of money and bonds is achieved by assuming cash-in-advance constraint. Agents are subject to idiosyncratic one period marginal utility shock, and trade takes place in a competitive market. As people can adjust their portfolios after the shock realizes, agents with higher marginal utility can effectively borrow from agents with lower marginal utility by transferring purchasing power from the former to the latter using illiquid bonds. This implicit borrowing and lending enabled by illiquid bonds improves welfare in Kocherlakota (2003)[1]. If bonds can be readily used as a payment as money, people will hold both assets only when there are no interest on bonds, so bonds are equivalent to money. There are two main differences that distinguishes mine from Kocherlakota (2003). Although there is a idiosyncratic shock (producer and consumer status), which is similar to the preference shock in Kocherlakota (2003), I assume that people can adjust their asset holdings only before the shock realizes. More importantly, while it is assumed that bonds can be made illiquid in Kocherlakota (2003), it may not be a desirable assumption. If bonds are not illiquid per se, people have incentive to deviate from using only money when they trade. Even when social planner can force people to use only money for payment, it may not be optimal to do so. I address this issue by considering a class of trading mechanism, under which people can jointly defect in pairwise trade.

Hu and Rocheteau (2013) uses an economy that builds on Lagos and Wright (2005), where they extend the setting by adding physical capital. Physical capital plays double duty in that economy, as an input for production in one subperiod and means of payment in another subperiod. Hence, physical capital can be accumulated too much to supplement money as a medium of exchange. Like mine, they also follow the mechanism design approach and consider a class of trading mechanisms, and characterize the optimal one. Under the optimal trading mechanism, physical capital have higher rate of return than money to prevent the overaccumulation of capital. So, coexistence of money and assets with higher return (physical capital) is necessary to achieve welfare improvement. As I consider intrinsically useless bonds rather than physical capital as higher return assets, such inefficiency does not arise here. I investigate a distinctive channel through which higher return assets improve social welfare.

[1] Andolfatto (2011); Boel and Camera (2006); Shi (2008) show that the result persists in steady state. See also Kim and Lee (2009).

2 The Model

The background setting is a variant of Shi (1995) and Trejos and Wright (1995). Time is discrete and infinite. There are nonatomic measure of people, who live forever and maximize expected lifetime utility with discount factor $\beta \in (0,1)$. People are anonymous, so their histories are private information and they cannot commit to future actions. In each period, agents sequentially enter to portfolio choice stage and pairwise trade stage.

All production and consumption occur in pairwise trade stage. In this stage, people randomly meet in pairs and the period status, a consumer, a producer, or inactive, is determined randomly according to the parameter $\mu \in \left(0, \frac{1}{2}\right)$. One becomes a producer or a consumer with probability μ respectively, and inactive with probability $1-2\mu$. Underlying structure of this specification is in Shi (1995) and Trejos and Wright (1995) (see also Williamson and Wright (2010)). Period utility function of an agent is $u(y) - x$, where $y,\ x \in \mathbb{R}_+$ are amount of consumption and production respectively. As one cannot produce and consume in a same period, at most one of y and x is strictly greater than zero in a period for an agent. $u(y)$ is strictly increasing, strictly concave, differentiable, and satisfies $u(0) = 0$. Also, $\hat{y} \equiv \max_{y \geq 0} [u(y) - y]$ is strictly positive. All produced goods are perishable. As no record-keeping is feasible in this economy, one cannot use any form of credit for trades. As a result, it is not possible to achieve any production and consumption without a medium of exchange.

There are two kinds of intrinsically useless and indivisible assets which can be used as a medium of exchange. I will call the first asset "currency"[2] and the second asset "bonds". These assets are fully recognizable and are not counterfeitable. An agent can hold assets up to 2 units in sum at any point of time. The set of feasible individual portfolio is

$$\mathbb{Z} \equiv \left\{ (M_C, M_B) \in \{0,1,2\}^2 \mid M_C + M_B \leq 2 \right\} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$$

In portfolio choice stage, agents can redeem their bonds for currency at one to one rate, and they can purchase bonds using currency also at one to one rate through a window that social planner operates. While this window can potentially be more sophisticated, by differentiating

[2] I call it "currency" rather than "money" to be clear.

Figure 2.1: Sequence of Actions.

exchange rate depending on the asset holdings, one virtue of this window is that it can be implemented by a competitive market. I assume that bonds mature in one period and are redeemed only in the following period[3]. One crucial (and the only meaningful) difference between two assets is that currency may disintegrate at the end of a period while bonds do not, in the following sense.

At the end of a period, people who are not at the upper bound of assets holding[4] receive an additional unit of currency with probability $\epsilon(z) \in [0,1]$, where the rate can depend on the asset holdings. After the transfer, each unit of currency disintegrates with probability $\tau \in [0,1]$. Hence, an agent with j unit of currency will lose all of her currency with probability τ^j, $j-1$ unit of currency with probability $(1-\tau)\,\tau^{j-1}$ and so on. This process is our stand-in for currency transfer and inflation, and they are modeled in this way to cope with the upper bound on assets holdings. This technique was also used in Li (1995), Deviatov and Wallace (2001) and Deviatov (2006). Bonds do not suffer from this disintegration process, and this difference makes bonds a higher return asset. For simplicity, I only consider such inflation-indexed bonds. At least conceptually, it is easy to accommodate bonds with different interest rates in this setup. Lastly, two rates are chosen by social planner as policy variables.

3 Implementable Allocations

Before defining the implementability, I describe information structure and the notion of defection. Table 1 shows the possible specifications in each stage, where a cell is a combination of the allowed defection (row) and the information structure (column). The specification that I use is marked with X. I assume that people in a pair can defect jointly (group defec-

[3]It is equivalent to assume that bonds can be redeemed at any periods and the date of issuance is disregarded.

[4]For example, if an agent has one unit of each asset, he cannot get the transfer since he is at the upper bound already. This assumption is made due to the upper bound on the asset holdings and for simplicity. In this sense, the scheme cannot perfectly resemble lump-sum transfer.

Table 1: Information structure (column) and notion of defection (row).

Trade stage

defection \ info.	sym	asym
individual		
group	X	

Transfer stage

defection \ info.	sym	asym
individual		X
group		

tion), in the sense that they can reject the designated trade by the planner if it is not in the pairwise core. This assumption puts more discipline on social planner, as the trading mechanism must exploit all the gains from trade. There is no private information in trade stage (symmetric information), hence the pairwise core is well-defined in that stage.

People cannot jointly defect in transfer stage (individual defection). An implication of this assumption is that they cannot pool their assets to exploit the transfer from the planner. While they cannot overstate their asset holdings (it can be verified simply by asking to show their assets), they can understate (asymmetric information). Hence, transfer policy is incentive compatible only when it is an increasing function of asset holdings.

I focus only to stationary allocations due to tractability. Hence, all time subscripts are omitted. Briefly speaking, implementable allocations satisfy three conditions in addition to stationarity: (1) *optimal portfolio choice*, (2) *coalition-proofness*, (3) *incentive compatibility of transfer policy.*

I will denote the expected discounted utility for an agent with portfolio z who enters the pairwise trade meeting stage by $v(z)$. The wealth (the sum of currency and bonds) distribution at the portfolio choice stage is $\{\pi_k\}_{k\in\{0,1,2\}}$. As we don't need to distinguish currency and bonds when people are entering this stage, it suffices to track the wealth distribution here. Transition probability from wealth level k to portfolio z at the portfolio choice stage is $t^{(1)}_{zk}$. All variables are listed in the table 2.

In portfolio choice stage, each agent chooses a portfolio that maximizes his expected utility. *Optimal portfolio choice condition* is satisfied when

$$z' \in \underset{\{x\in\mathbb{Z}|x_C+x_B\leq k\}}{\arg\max}\ v(x)\ \text{, if } t^{(1)}_{z'k} > 0 \tag{3.1}$$

For given $t^{(1)}$and π_k, the measure of people with portfolio z at trade stage $p^{(1)}_z$ is

Table 2: Notations.

Symbol	Description
$\epsilon(z) \in [0,1]$	Transfer rate
$\tau \in [0,1]$	Disintegration rate
$t^{(1)} : \{0,1,2\} \to \Delta(\mathbb{Z})$	Transition Probability (Portfolio Choice Stage)
$t^{(2)} : \mathbb{Z} \to \Delta(\mathbb{Z})$	Transition Probability (Trade Meeting Stage)
$t^{(3)} : \mathbb{Z} \to \Delta(\{0,1,2\})$	Transition Probability (Transfer and Disintegration)
$v : \mathbb{Z} \to \mathbb{R}_+$	Expected Discounted Utility (Pre Trade Stage)
$w : \mathbb{Z} \to \mathbb{R}_+$	Expected Discounted Utility (Post Trade Stage)
$p^{(1)} \in \Delta(\mathbb{Z})$	Portfolio Holding Dist. (Pre Trade Stage)
$p^{(2)} \in \Delta(\mathbb{Z})$	Portfolio Holding Dist. (Post Trade Stage)
$\pi \in \Delta(\{0,1,2\})$	Wealth Distribution (Portfolio Choice Stage)
$y : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$	Production Level (Trade Stage)
$\lambda_S : \mathbb{Z} \times \mathbb{Z} \to \Delta(\mathbb{Z})$	producer's Portfolio Dist. (Post Trade Meeting)
$\lambda_B : \mathbb{Z} \times \mathbb{Z} \to \Delta(\mathbb{Z})$	consumer's Portfolio Dist. (Post Trade Meeting)
$\gamma : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$	producer's Surplus (Trade Meeting)

$$p_z^{(1)} = \sum_{k \in \{0,1,2\}} t_{zk}^{(1)} \pi_k$$

Given that there are 6 possible portfolios that consumers and producers can hold, we have 36 types of different meetings. Each type of meeting is a combination of the portfolios of the producer and the consumer $(z_p,\ z_c)$. Denote the production level in each meeting by $y(z_p, z_c)$, and let $\lambda_c(z_c'; z_p, z_c)$, $\lambda_p\left(z_p'; z_p, z_c\right)$ be the distribution of the portfolios of the consumer and the producer, as a result of the trade meeting. Payments of currency and bonds are embedded in this representation, since those distributions can be derived from a given payment of currency and bonds, and vice-versa. In sum, transition probability from portfolio z to portfolio z' at the trade stage $t_{z'z}^{(2)}$ is characterized by the probability of entering to each type of meeting and having a certain portfolio after that meeting.

$$t_{z'z}^{(2)} = \begin{cases} \mu \sum_x p_x^{(1)} \left[\lambda_S(z'; z, x) + \lambda_B(z'; x, z)\right] & \text{if } z' \neq z \\ 1 - \sum_{z' \neq z} t_{z'z}^{(2)} & \text{if } z' = z \end{cases}$$

The distribution of agents over portfolios after the trade meeting stage, $p^{(2)}$, is

$$p^{(2)}_{z'} = \sum_{z\in\mathbb{Z}} t^{(2)}_{z'z} p^{(1)}_z$$

The currency transfer rate $\epsilon_z \in [0,1]$ and the disintegration rate $\tau \in [0,1]$, which are chosen by the social planner, determine the transition probability from portfolio z to wealth level m at transfer and disintegration stage $t^{(3)}_{mz}$. For example, if an agent holds one unit of currency, she will end up with one unit of currency with probability $t^{(3)}_{(10,10)} = \epsilon_{10}\tau + (1-\epsilon_{10})(1-\tau)$, where the first term represents the probability of receiving one unit of currency and losing it, and the second term represents the probability of not receiving nor losing. Note that agents are allowed to hide their asset holdings. So, they will misrepresent their asset holdings if the transfer scheme is not incentive compatible. The transfer policy is *incentive compatible* if and only if

$$\epsilon_z \geq \epsilon_{z'} \text{ whenever } z \geq z' \tag{3.2}$$

The wealth (the sum of currency and bonds) distribution at the portfolio choice stage in the next period satisfies

$$\pi'_k = \sum_{z\in\mathbb{Z}} t^{(3)}_{kz} p^{(2)}_z \text{ for } k \in \{0,1,2\}$$

Two value functions, before and after trade meeting stage, are

$$\begin{aligned}
v(z) &= \mu \sum_{x\in\mathbb{Z}} p^{(1)}_x \left\{ u\left[y(x,z)\right] + E_{\lambda_C(x,z)} w - y(z,x) + E_{\lambda_P(z,x)} w \right\} \\
&\quad + (1-2\mu)\, w(z) \\
w(z) &= \beta \sum_{x\in\mathbb{Z}} \sum_{m\in\{0,1,2\}} t^{(3)}_{mz} t^{(1)}_{xm} v'(x)
\end{aligned}$$

for all $z \in \mathbb{Z}$, where

$$
\begin{aligned}
E_{\lambda_C(x,z)} w &\equiv \sum_{z'_C} \lambda_C \left(z'_C ; x, z \right) w \left(z'_C \right) \\
E_{\lambda_P(x,z)} w &\equiv \sum_{z'_P} \lambda_P \left(z'_P ; x, z \right) w \left(z'_P \right)
\end{aligned}
$$

Lastly, stationarity requires

$$
\begin{aligned}
\pi'_k &= \pi_k \text{ for all } k \in \{0, 1, 2\} && (3.3) \\
v'(z) &= v(z) \text{ for all } z \in \mathbb{Z} && (3.4)
\end{aligned}
$$

While many works using this class of models adopt some kind of bargaining solution to determine production levels and payments in trade meeting, I adopt mechanism design approach as described in Wallace (2010), which considers all implementable allocations. Using this approach has two important implications. First, it enables to achieve coexistence of currency and bonds in the spirit of Zhu and Wallace (2007). As two assets have no differences in physical properties (i.e. recognizability, storage costs among many others) other than their labels and vulnerability to disintegration, agents will not hold currency if trade outcomes are determined by a bargaining solution under which trade outcomes depend only on the feasible allocations and termination values. Second, it eliminates the loss attributable to sub-optimal trading rule. I aim to show that bonds can still improve welfare even when the welfare loss due to sub-optimal trading rule disappears.

Production and payment are required to maximize consumers' utility subject to making producers at least better than no trade. $\gamma(z_P, z_C)$ is the surplus of producers in each type of meeting. Formally, for any given w, $\{y(\cdot,\cdot), \lambda_C(\cdot;\cdot,\cdot), \lambda_P(\cdot;\cdot,\cdot)\}$ are *coalition-proof* if for every z_P and z_C, there exists $\gamma(z_P, z_C) \geq 0$ such that $\{y(z_P, z_C), \lambda_C(z_P, z_C), \lambda_P(z_P, z_C)\}$ is a solution to the following problem:

$$
\begin{aligned}
\max_{x\in\mathbb{R}_+,\eta_C\in\Delta(\mathbb{Z}),\eta_P\in\Delta(\mathbb{Z})} \quad & u(x)+E_{\eta_C}w-w(z_C) \\
\text{subject to} \quad & -x+E_{\eta_P}w-w(z_P)\geq\gamma(z_P,z_C) \\
& u(x)+E_{\eta_C}w-w(z_C)\geq 0 \\
& \eta_C(z_P+z_C-z')=\eta_P(z') \text{ for } z'\in\{(m,b)\,|0\leq(m,b)\leq z_P+z_C\} \\
& \eta_P(z')=\eta_C(z')=0 \text{ for } z'\notin\{(m,b)\,|0\leq(m,b)\leq z_P+z_C\}
\end{aligned}
$$

Now, we can define implementable allocations.

Definition 1. An allocation $p^{(1)}\in\Delta(\mathbb{Z})$, $y:\ \mathbb{Z}\times\mathbb{Z}\to\mathbb{R}_+$, and $\lambda:\ \mathbb{Z}\times\mathbb{Z}\to\Delta(\mathbb{Z})\times\Delta(\mathbb{Z})$ is *implementable* if there exist ϵ, τ, v, π, $t^{(1)}$ that satisfy *optimal portfolio choice condition (3.1), incentive compatibility of transfer policy (3.2), stationarity (3.3), (3.4) and coalition-proofness.*

As standard in the literature, I use the ex-ante utility level, where the initial portfolios of agents are randomly assigned according to the stationary distribution, as the welfare criterion.

$$
W\equiv\sum_z p_z^{(1)}v(z)
$$

Maximizing it is equivalent to maximizing one period total surplus

$$
\sum_z\sum_{z'}p_z^{(1)}p_{z'}^{(1)}\left[u(y_{zz'})-y_{zz'}\right]
$$

An optimum achieves the highest welfare among all the implementable allocations.

Under this notion of the implementability, the set of implementable allocations in an economy with only currency (henceforth, currency economy) is nested to the set in an economy with currency and bonds (henceforth, bond economy). In other words, adding bonds to an economy is always weakly better. Pick an arbitrary implementable allocation in currency economy. To implement the same allocation in bond economy, it suffices to restrain agents from choosing bonds in portfolio choice stage, and that can be done by the following trading mechanism. Suppose that agents are asked to produce and pay as before in any meetings in which people hold only currency. In meetings where only one person holds bonds, the bond holder gets no gains from trade and the meeting partner gets all gains. If this is the case, given that no one else holds bonds, an agent will not choose to hold bonds at the portfolio choice stage. Since production and payment are the same as before, this trading mechanism implements a same allocation in bond economy. This nesting argument relies on two assumptions that I made. First, if the bond earns higher interest, agents may sacrifice the period surplus for the sake of the interest. In this case, social planner is not able to prevent people from holding bonds as above. Second, this argument is not possible if agents' portfolios are not visible in trade stage.

Remark. An implementable allocation in currency economy is also implementable in bond economy.

Before moving on to the next section, I want to make and justify an assumption on the meetings where producers already have two units of assets. When a producer is already at the upper bound of asset holdings with some currency (1 unit of each asset, or 2 units of currency), he may be willing to produce to earn bonds by swapping assets. Suppose that a producer with two units of currency meets a consumer with one unit of bond. Even though the producer has two units of assets already, he may be willing to produce to exchange his currency to a bond. In the following, I assume that this kind of trade cannot happen.

Assumption. *If a producer has two units of any assets, he cannot alter his portfolio by exchanging assets in trade stage.*

By making this assumption, I want to shut down one channel through which bonds can improve the welfare in this class of model, a reminiscent of Aiyagari et al. (1996). Hence, welfare improvement after making this assumption will be attributed to a different channel. In the appendix, I explain the rationale of the assumption using a same economy with a one-unit upper bound instead of a two-unit upper bound on asset holdings. I construct a numerical example in which adding bonds improves welfare by increasing the number of trade meetings, similar to Aiyagari et al. (1996), and then prove that this improvement disappears once asset swapping is not allowed. Hence, adding another asset in a two-unit upper bound economy is not a mere extension of the one in a one-unit upper bound economy. With richer set of asset holdings, we can see a different channel of welfare improvement from adding bonds.

4 Numerical Examples

In this section, I use numerical examples to learn about a beneficial role of government bonds. I use same utility function and parameters in Deviatov (2006), which are arbitrary except for the discount factor β. Define the first-best as people produce and consume $y^* \equiv \max_{y\geq 0}\left[u\left(y\right)-y\right]$ in every trade meeting. If perfect monitoring is possible, the first-best allocation is implementable whenever β satisfies

$$\frac{u\left(y^*\right)}{y^*} \geq \frac{1-\beta}{\beta\mu}+1$$

The discount factors that I used satisfy above inequality, so that the first-best allocation would be implementable under perfect monitoring.

I first compute the optimum in currency economy. The optimum in this economy provides the upper bound of welfare that the planner can achieve using only currency. The only difference between this computation and Deviatov (2006) is that transfer can now depend on asset holdings. Then, I formulate the maximization problem of social planner in bond economy, and find a better allocation. For simplicity, I assume that transfer rate is the same regardless of asset holdings. This assumption is innocuous for my purpose as it restricts what government can do in bond economy. I use GAMS (General Algebraic Modeling System) and BARON (Branch-And-Reduce Optimization Navigator) solver to compute these problems. GAMS is a modeling system for mathematical programming and optimization, and BARON solver can be used in this system. BARON solver uses deterministic global optimization algorithms of the branch-and-bound type, and it is one of the most robust global solver according to Neumaier et al. (2005). While the program did not complete the computation for the bond economy[5], I could find a better allocation than the optimum in currency economy. In the course of finding the global solution, the solver constantly updates and records its

[5] It is known that global solvers are generally slower than local solvers, but other alternatives are not more suitable for my purposes.

candidate solution, and the candidate is an implementable allocation. By comparing that allocation to the optimum in currency economy, we can learn how government can use bonds to improve welfare.

I use $u(y) = y^{0.2}$, the coincidence parameter $\mu = \frac{1}{3}$, and two discount factors $\beta \in \left\{\frac{1}{2}, \frac{2}{3}\right\}$. Under this specification, the first-best production level is $y^* \approx 0.1337$. In Deviatov (2006), he finds positive transfer and inflation optimal for the lower discount factor ($\beta = \frac{1}{2}$), while no transfer and zero inflation optimal for the higher one ($\beta = \frac{2}{3}$). It turns out that allowing different transfer rate for different asset holdings does not change the optimum for these examples, as the optimum in currency economy shows same transfer rate for different asset holdings. The computed optimum is same as the optimum in Deviatov (2006)[6].

Table 3 shows welfare and aggregates of the optimum in currency economy and a better allocation in bond economy, when discount factor β is $\frac{1}{2}$. The welfare level, relative to the first-best one, is in the first row. The second and third row show transfer and disintegration rate in each allocation. Transfer rate is only one number as optimal transfer rate is same for different asset holdings in currency economy, and it is assumed to be same in bond economy. They are strictly positive in both allocations, but transfer rate and disintegration rate are higher in the bond allocation. This change is consistent with the change in distribution (π), which is in the last row. Each number in the cell of that row shows the proportion of people with 0,1,2 units of assets respectively, in the beginning of each period. In the bond allocation, agents with 2 units of assets choose one of each asset at portfolio choice stage. Agents with 1 unit of asset choose to hold currency. Hence, there are 4 types of trade meeting that happen on equilibrium in each economy, and we can compare those meetings across the economies.

Each cell in table 4 contains production and payment in a meeting in which the producer and the consumer have portfolios corresponding to the row and the column. For the portfolios in bond economy, I use the first digit for currency and the second digit for bonds. The first number in each cell shows the production level relative to first-best, and the number(s) in the right is the amount of assets paid from the consumer to the producer. Note that

[6] While it is out of scope of this paper, an example showed that incentive compatibility of transfer policy is binding. When asset holdings are not private information at transfer stage, we can disregard the incentive compatibility of transfer policy. In this case, the optimum of currency economy did not show same rate for different asset holdings, and welfare was strictly higher than the one with equal transfer rate.

Table 3: Welfare and aggregates ($\beta = 0.5$).

	Currency Optimum	A Better Bond Allocation
$\frac{W}{W^*}$	0.419	0.426
$\epsilon(\%)$	0.250	0.472
$\tau(\%)$	0.176	0.346
π	0.369 / 0.397 / 0.234	0.346 / 0.418 / 0.237

Table 4: Output (relative to the first best level) and payment in trade stage ($\beta = 0.5$).

Currency Optimum

$P\backslash C$	1	2
0	0.435 / 1	0.435 / 1
1	0.120 / 1	0.120 / 1

A Better Bond Allocation

$P\backslash C$	1,0	1,1
0,0	0.282 / 1,0	0.431 / 0,1
1,0	0.103 / 1,0	0.206 / 0,1

it is not optimal to randomize over production as the utility function is concave and cost function is linear. It must be deterministic to satisfy coalition-proofness constraint. Although lotteries for payments are potentially useful, they are not used in both allocations. In all of meetings that occur on equilibrium, consumers get all the surplus of trade and producers become indifferent to no trade. Hence, the trading mechanism is equivalent to the one where consumers make take-it-or-leave-it offers in those meetings. Needless to say, that is not the case for the meetings that do not occur on equilibrium. Production level is higher in (10,11) meeting, where the first element indicates the producer's portfolio, than the comparable (1,2) meeting in currency economy. However, production levels in all other meetings are lower. In a sense, the production is smoothed over different types of meetings. In all of meetings in both economies, one unit of asset is transferred with certainty (numbers in the parentheses). When a consumer has bonds, one unit of bond is paid. Otherwise, he pays with one unit of currency.

One may wonder why the output level of (00,11) meeting in bond economy is lower than the output of (0,2) meeting in currency economy (first row, second column). The producer in bond economy gets a bond, which does not suffer from disintegration, while his counterpart in currency economy gets a unit of currency. One reason is that currency has lower value in the bond allocation. Because of the higher transfer and inflation in the bond allocation, the return on currency decreases. As the value of bonds is tied to the value of currency, it can result in lower output level. In contrast, the output level in (10,11) meeting is higher than

Table 5: Welfare and aggregates ($\beta = \frac{2}{3}$).

	Currency Optimum	A Better Bond Allocation
$\frac{W}{W^*}$	0.478	0.487
$\epsilon(\%)$	0	0.057
$\tau(\%)$	0	0.176
π	0.323 / 0.430 / 0.247	0.316 / 0.477 / 0.207

Table 6: Output (relative to the first best level) and payment in trade stage ($\beta = \frac{2}{3}$).

Currency Optimum

$P \backslash C$	1	2
0	1 / 0.587	1.703 / 1
1	0.212 / 0.431	0.493 / 1

A Better Bond Allocation

$P \backslash C$	0,1	1,1
0,0	1 / 0, 0.656	1.127 / 1,0
0,1	0.139 / 0, 0.442	0.257 / 1,0

the one in (1,2) meeting. That can be explained by the concavity of value functions. If we compare the two value functions in currency optimum and the bond allocation ($v^C(\cdot)$ and $v^B(\cdot)$),

$$\begin{aligned} v^C(1) - v^C(0) &\geq v^B(10) - v^B(00) \\ v^B(11) - v^B(10) &\geq v^C(2) - v^C(1) \end{aligned}$$

For an agent who already has a unit of currency, attaining an additional asset is more worthwhile in the bond allocation. As a result, the output level in (10,11) meeting is higher than the one in (1,2) meeting.

I find a similar result with a higher discount factor, $\beta = \frac{2}{3}$. In this example, the optimal transfer and inflation rate is zero in currency optimum. But, a better allocation can be implemented with bonds, positive transfer and inflation. Now, agents with one unit of asset choose a bond, while agents with two units of asset still choose one unit of each asset. The pattern of aggregates is similar to the previous one: higher transfer, disintegration, and more people in the center of the distribution. One thing to note is that the magnitude of transfer rate is smaller than the disintegration rate, while it was the opposite before. As more people hold bonds now, higher inflation follows even with small amount of currency creation.

As before, consumers get all the surplus of trade in meetings that occur on equilibrium. Production level decreases in all meetings that actually happen. So, the improvement on

Table 7: Expected utility (relative to first-best level) for each wealth level.

$\beta = \frac{1}{2}$

	Currency optimum	The Bond allocation
0	0.0987	0.1273
1	0.5776	0.5395
2	0.653	0.6637

$\beta = \frac{1}{2}$

Currency optimum	The Bond allocation
0	0.0553
0.6387	0.6492
0.8234	0.7731

welfare is mostly attributable to the improved extensive margin. When consumers have only one asset, they use lottery for payment in both economies. In addition, people in the bond allocation pay different assets from previous example. Consumers with two units of assets pay one unit of currency now, while consumers with one unit of asset pay a bond.

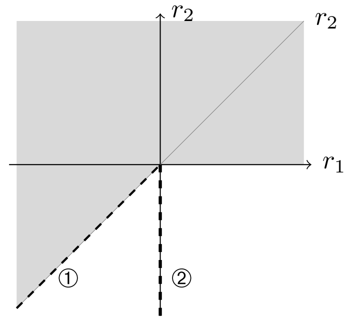
So how do bonds improve the welfare in this economy? The optimal rate of transfer and disintegration balance the trade-off between *beneficial effect* on the extensive margin (number of trade meeting) and *harmful effect* on the producer's incentive to produce for getting currency. Lump-sum transfer and inflation is beneficial on the extensive margin, as they alter the asset holdings distribution in a way that makes use of more trade opportunities. If a producer has two units of assets, or a consumer has none, that trade opportunity will be wasted. Some lump-sum transfer and inflation are helpful to prevent such a loss. However, they also decrease the return on currency, so that a producer would be willing to produce less for getting currency. If the *harmful effect* on producers' incentive can be mitigated, higher welfare can be achieved through higher rate of currency transfer. Note that wealthy agents loses more when τ increases. Hence, by inducing wealthy agents to hold one unit of bond, social planner can mitigate the incentive effect in more sophisticated way. In consequence, the bond allocation results in higher welfare by having higher transfer rate (and disintegration rate), while the presence of bonds mitigate the loss to wealthy agents resulting from this increase in transfer and inflation rate. The change in distribution is in line with this argument.

Comparison of v provides a different account of the welfare gain. In the first example, people with 0 wealth level and 2 wealth level are better off, while agents with 1 unit of wealth are not in the bond allocation. Note that 0 wealth agents can only be a producer, who earns no trade surplus in trade stage in both allocations. However, he has a higher chance of having 1 wealth, due to higher rate of currency creation. 2 wealth agents are better off since they can induce more production from 1 wealth agents, and the loss from disintegration is mitigated by holding bonds. On the contrary, high currency creation rate and disintegration rate is not beneficial to 1 wealth agents since the gain from transfer (becoming 2 wealth agents by earning free currency) is not compensating the loss (becoming 0 wealth agents due to disintegration). Also, he consumes less due to lower value of currency in this economy. The second example shows a different pattern. While people with 0 wealth still get better in the bond allocation, changes in expected utility for people with 1 and 2 wealth are in contrast to the first example. People with 1 wealth get better due to the chance of earning free currency and the changes in distribution, while disintegration does not decrease the value of their payment (bonds). Even though 2 wealth agents in the bond allocation hold a unit of bond, he (possibly) loses currency due to disintegration, while they don't in currency optimum as the disintegration rate is zero. In addition, they consume less as their payment has lower value. While they could use a bond for payment, they choose to retain it for not putting themselves at risk of having no asset due to disintegration.

Based on this explanation, it is reasonable to conjecture that the beneficial role of bonds will disappear for some parameters. In Deviatov (2006), positive transfer and inflation is not beneficial for a range of parameters (high discount factor, low risk aversion), as the harmful effect dominates the beneficial effect for small transfer and inflation. As the role of bonds is associated with the beneficial effect of transfer and inflation, I expect that the optimum will not make use of bonds under high discount factor and low risk aversion. Verification of this conjecture requires finding the optimum with two assets, which is not feasible as of now due to computational limitations. However, the program did not find any better allocations than the currency optimum even after considerable computation time for a high discount factor ($\frac{1}{1.05}$) case, and this is consistent with the conjecture.

Figure 4.1 illustrates the region that Deviatov (2006) and this paper investigated. Consider an economy where people are anonymous and cannot be taxed as in this paper. Each axis represents the (real) rate of return of a fiat asset, and think of a point as an economy with two fiat assets with returns corresponding to the point. Since fiat assets are intrinsically worthless, it is impossible to support any points in first quadrant including the axes in that

Figure 4.1: Real interest rates of assets.



region without tax. Also, all points above 45 degree line are mirror-image of points in the region below the line. So the gray area is not of interest. Deviatov (2006) considers the region on the 45 degree line, currency only economy, excluding the aforementioned regions, and the area is ①. This paper expands the region to include negative y-axis, so ① and ②. Points between those two lines can be studied by making bonds partially indexed to inflation rate. Some points in the fourth quadrant may be achievable by paying higher interest rate than inflation rate, but not all of them are. For example, if we think of a point (x, y) in the fourth quadrant with high x and y close to zero, inflation rate should be close to zero while the real interest rate on bonds is very high. Since they are intrinsically worthless assets, it is infeasible to support such a high return without causing high inflation. In general, if we can define a feasible region, the planner would not choose a point on the lines that are studied here. I leave for future work the study of the optimal rate of returns.

5 Concluding Remarks

Should two forms of government liabilities - currency and government bonds - exist? Friedman (1948) argues that government should get rid of interest bearing government liabilities and finance its expenditures through money creation and tax. Although there already exist a number of papers on societal benefits of interest bearing assets following Kocherlakota (2003), this paper differentiates itself from the previous ones as following. First, coexistence is achieved without relying on any assumed advantages of currency over bonds used. Second, the interest bearing asset still improves welfare after eliminating the loss from sub-optimal trading mechanism. Lastly, the role exists only when money creation is potentially useful[7].

As I have presented only two examples, one may wonder if the result holds more generally. It is limited in the sense that assets are indivisible and there is a somewhat small upper bound on assets holding. Applying other parameters can be easily done, but it does not seem to add much as the contribution of this paper is the discovery of new channel through which bonds improve social welfare. I suspect that the result would persist with divisible and unbounded asset holdings as well. Suppose there are currency and inflation-indexed bond, and social planner is able to choose the transfer (in a lump-sum fashion) rate of currency. Social planner will still have some power, though limited, to induce wealthy agents to choose a portfolio with more of indexed bond, by appropriately dividing trade surplus contingent on portfolios. This will create another policy dimension, and the presence of inflation-indexed bond can implement an allocation which was not implementable before. If money to be essential, we ought to put ourselves in a world where first-best is not achievable (See Wallace (2010)). The additional fiat asset with higher return helps to get closer to the first-best.

Other crucial assumptions are the complete information of portfolios in trade meetings and the size of pairwise meeting. When the portfolios of a consumer and a producer is not common knowledge, we need to use a different notion of pairwise core, core under incomplete information. It is expected that the choice of the notion would affect the result. Regarding the size of meetings, consider the other extreme case in which people meet altogether. In this case, the only core allocation is the competitive equilibrium allocation. Since social planner uses multiplicity of core to induce agents to hold a particular portfolio, what the planner can do will be more limited.

Features of the optimum is also of interest. Due to computational limitations, I find a welfare superior allocation instead of an optimum. Optimal transfer, inflation rate and trading mechanism may give some further insights to understand the beneficial role of government bonds.

[7] Wallace (2014) conjectures that there generically exists beneficial government money creation in a class of monetary economies

A Appendix

In this appendix, I change the set of feasible asset holdings to have a one-unit upper bound while keeping environment and solution concepts the same.

$$\mathbb{Z} \equiv \left\{ (M_C, M_B) \in \{0,1\}^2 \mid M_C + M_B \leq 1 \right\} = \{(0,0), (0,1), (1,0)\}$$

Example. This example is two implementable allocations in an economy with a unit upper bound. First one is the optimum of currency economy, and the other one is a welfare superior allocation in bond economy, compared to the optimum in currency economy.

The optimum is depending on the following inequality in the economy with $\{0,1\}$ money holding.

$$\frac{u(y)}{y} \geq \frac{1-\beta}{\mu\beta}\frac{1}{1-m} + 1$$

If $y = \hat{y}$, where $\hat{y} \equiv \max_{y \geq 0} [u(y) - y]$, and $m = \frac{1}{2}$ satisfies the inequality, it is the optimum. If not, optimal production y^o and amount of money m^o holds above inequality with equality and $y^o < \hat{y},\ m^o < \frac{1}{2}$. Using the equality, social planner's maximization problem

$$\max_{m \in [0,1], y \geq 0} W^{(1)} \equiv m(1-m)(u(y) - y)$$

becomes

$$\begin{aligned} \max_{y \geq 0} \quad & \left(\tfrac{1-\beta}{\beta\mu} y\right)\left(1 - \tfrac{1-\beta}{\beta\mu}\tfrac{y}{u(y)-y}\right) \\ \text{subject to} \quad & m = 1 - \tfrac{1-\beta}{\beta\mu}\tfrac{y}{u(y)-y} \in [0,1] \\ & 1 - m = \tfrac{1-\beta}{\beta\mu}\tfrac{y}{u(y)-y} \in [0,1] \end{aligned}$$

Using the first parameter in section 4, we get $W^{(1)} = 0.1319$, $y = 0.0909$, $m = 0.4839$.

Now, construct an allocation in the bond economy as follows. Assume that $m_C = m_B = \frac{1}{4}$, disintegration rate $\tau = 0.01$ (the transfer rate that preserves stationarity is $\epsilon = \frac{0.01*0.25}{0.99*0.5}$) and buyers make take-it-or-leave-it offers in (0,C), (C,B) meeting, where two elements are the asset holdings of a producer and a consumer respectively. Also, coexistence requires that agents are indifferent to choose currency or bonds at the portfolio choice stage. Following expressions are the value functions of agents holding each asset before and after trade meeting stage.

$$\begin{aligned}
V_C &= \mu(1-m_C-m_B)(u(y_C)+\beta W_0)+\mu(m_B)(-y_{CB}+\beta W_B)+(1-\mu(1-m_C))\beta W_C \\
V_B &= \frac{1-m_C-m_B}{N}(u(y_B)+\beta W_0)+\frac{m_C}{N}(u(y_{CB})+\beta W_C)+\left(1-\frac{1-m_B}{N}\right)\beta W_B \\
V_0 &= \mu m_C(-y_C+\beta W_C)+\mu m_B(-y_B+\beta W_B)+(1-\mu(m_B+m_C))\beta W_0 \\
W_C &= (1-\tau)V_C+\tau V_0 \\
W_B &= V_B \\
W_0 &= \epsilon(1-\tau)V_C+(1-\epsilon+\epsilon\tau)V_0
\end{aligned}$$

As buyers make take-it-or-leave-it offers in (0,C), (C,B) meeting,

$$\begin{aligned}
-y_C+\beta(W_C-W_0) &= 0 \\
-y_{CB}+\beta(W_B-W_C) &= 0
\end{aligned}$$

and these are equivalent to

$$\begin{aligned}
y_C &= \beta(1-\epsilon)(1-\tau)(V_C-V_0) \\
y_{CB} &= \beta\tau(V_C-V_0)
\end{aligned}$$

using $V_C = V_B$ (coexistence condition) . After some algebra, $V_C = V_B$ implies

$$(1-m_C-m_B)(u(y_C)-u(y_B))=\frac{m_C+m_B}{2}(u(y_{CB})+y_{CB})+\left(\frac{1}{\mu}-1\right)y_{CB}$$

Lastly, $V_C - V_0$ can be expressed as

$$\begin{aligned}
(V_C-V_0) &= \mu(1-m_C-m_B)u(y_C)+\mu m_C y_C+m_B(y_B-y_{CB}) \\
&\quad +\beta(1-\mu)(1-\epsilon)(1-\tau)(V_C-V_0)
\end{aligned}$$

One solution that satisfies above four equations is $y_C = 0.0802$, $y_B = 0.0253$, $y_{CB} =$

0.000814 . The welfare level of this allocation

$$W^{(2)} \equiv \sum_{i=B,C} m_i \left(1 - m_B - m_C\right) \left(u\left(y_i\right) - y_i\right) + m_B m_C \left(u\left(y_{CB}\right) - y_{CB}\right)$$

is $W^{(2)} = 0.1372 > 0.1319 = W^{(1)}$.

Hence, this allocation is welfare superior to the optimum in currency economy.

This example is a reminiscent of Aiyagari et al. (1996), where they show that there exists a better equilibrium with two different color of money, compared to the unique equilibrium with only one money. Even though two monies differ only in their appearances, one of them is more valuable in the better equilibrium. By valuing two colors of money differently, it endogenously generates a denomination structure, effectively a richer set of money holdings, and leads to welfare improvement. While the argument does not straightforwardly extend to the economy we consider, due to many differences between Aiyagari et al. (1996) and mine, above example shows that a similar result arises.

Next proposition proves that with a one-unit upper bound asset holdings, the welfare improvement disappears when asset swapping is not possible.

Proposition. *In a one-unit upper bound economy, bonds do not improve welfare when asset swapping is not possible.*

Proof. I show that for any given $m, \epsilon, \tau \in (0,1)$, where $m = m_C + m_B$, the set of implementable y_C in bond economy is a subset of the set of implementable y_C in currency economy. Since $y_C > y_B$ is necessary to achieve the coexistence whenever transfer and disintegration rate is strictly positive, this proves the claim. Consider following social planner's problem:

Given m, ϵ, τ

$$\begin{aligned} \max_{m_C \in [0,m], y_C, y_B} \quad & (1-m)\, m_C \left(u\left(y_C\right) - y_C\right) + (1-m)\left(m - m_C\right)\left(u\left(y_B\right) - y_B\right) \\ \text{subject to} \quad & -y_C + \beta W_C \geq \beta W_0 \\ & u\left(y_C\right) + \beta W_0 \geq \beta W_c \\ & -y_B + \beta W_B \geq \beta W_0 \\ & u\left(y_B\right) + \beta W_0 \geq \beta W_B \end{aligned}$$

where

$$\begin{aligned}
V_C &= \mu(1-m)(u(y_C)+\beta W_0)+(1-\mu(1-m))\beta W_C \\
V_B &= \mu(1-m)(u(y_B)+\beta W_0)+(1-\mu(1-m))\beta W_B \\
V_0 &= \mu m_C(-y_C+\beta W_C)+\mu(m-m_C)(-y_B+\beta W_B)+(1-\mu m)\beta W_0 \\
W_C &= (1-\tau)V_C+\tau V_0 \\
W_B &= V_B \\
W_0 &= \epsilon(1-\tau)V_C+(1-\epsilon+\epsilon\tau)V_0
\end{aligned}$$

By subtracting V_0 from V_C, we get

$$\begin{aligned}
\mu(1-m)u(y_C)+\mu(m_C y_C+(m-m_C)y_B) &= \Delta(1-\beta(1-\epsilon)(1-\tau)) \\
&\quad +\beta\mu\Delta((1-\epsilon)(1-\tau)+\tau(m-m_C))
\end{aligned}$$

where $\Delta \equiv V_C - V_0$. Using this, we can write the participation constraint of a producer who meets a consumer with currency as

$$\Omega(\mu(1-m)u(y_C)+\mu(m_C y_C+(m-m_C)y_B)) \geq y_C$$

where $\Omega = \frac{\beta(1-\epsilon)(1-\tau)}{1-\beta(1-\epsilon)(1-\tau)+\mu\beta((1-\epsilon)(1-\tau)+\tau(m-m_C))}$. This is equivalent to

$$(1-m)\frac{u(y_C)}{y_C}+m_C+(m-m_C)\frac{y_B}{y_C} \geq \mu\left(\frac{1}{\beta(1-\epsilon)(1-\tau)}-1\right)+\frac{\tau}{1-\tau}(1-\epsilon)(m-m_C)$$

The largest set of y_C that satisfies this necessary condition can be attained when $m_C = m$, and it is a subset of the set of implementable allocations in currency economy. □

Given the access to the window in every period, the only way to achieve the coexistence, in the sense that people are indifferent to hold currency or bonds, is by making currency buys more goods in trade stage than bonds, as bonds have an advantage in the disintegration process. While the extensive margin (number of trade meeting) does not increase by adding bonds as we forbid asset swapping, the intensive margin (amount of production and consumption) worsens in order to achieve the coexistence.

Welfare improvements in the example is associated with the upper bound and indivisibility on the feasible set of asset holding, while those assumptions are made solely to make the model tractable. As I do not want to focus on that issue, restrictions on such trades are in order. While above results are attained in an economy with a one-unit upper bound, it hints that we are turning ourselves away from the result in Aiyagari et al. (1996) by not allowing asset swapping.

One crucial difference between a one-unit upper and a two-unit upper bound is that inflation created by lump-sum transfer can be beneficial in a two-unit upper bound economy. The beneficial extensive margin effect can present only when people can hold more than one unit of asset. This observation helps to understand the difference between the proposition and the results in the section 4.

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